VARIATIONAL STATEMENT OF THE PROBLEM OF LIQUID MOTION IN A CONTAINER OF FINITE DIMENSIONS

(VARIATSIONNAIA FORMULIROVKA ZADACHI O DVIZHENII ZHIDKOSTI V SOSUDE KONECHNYKH RAZMEROV)

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Presented here is the variational statement of the problem of non-linear oscillations of the ideal, incompressible liquid enclosed in a container of finite dimensions and subjected to the forces of gravity and surface tension. The variational approach allows this problem to be solved by direct methods.

1. We shall consider the motion of the ideal, incompressible liquid in a container of finite dimensions. The liquid volume V partially fills the container, the remaining part is occupied by gas. It is assumed that the liquid is acted on by the gravitational forces of intensity g and the forces of surface tension. We shall denote the coefficient of surface tension for Σ_1 the container — gas interface by σ_1, the coefficient of surface tension for Σ_2 the container — liquid interface by σ_2, and σ, that, for S the gas — liquid interface.

We shall use the Cartesian system of coordinates $Oxyz$, with the plane xy normal to the vector $\mathbf{g}$ and the z-axis pointing vertically upward (Fig.1). Moreover, it is specified that the xy-plane coincides with the mean position of the liquid surface. S_0 shall denote the projection of free liquid surface S on this plane and L shall denote the intersection of the surfaces S and Σ the internal surface of the container. The elevation of the free surface above $z = 0$ shall be denoted by $\zeta(x,y,t)$. For the equilibrium position $\zeta = \zeta^\circ(x,y)$.

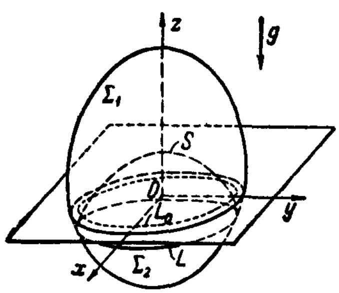

Fig. 1

We shall denote the velocity of the liquid by $\mathbf{v}$ and its density by ρ. We shall neglect the mass of the gas above the liquid and consider it to be quiescent.

2. The kinetic energy of the system is given by

$$T = \frac{1}{2}\rho \int_V v^2 dV \qquad (2.1)$$

Subject to above assumptions the potential energy can be written as

$$\Pi = \frac{1}{2}\rho g \int\limits_{S_0} \zeta^2\,dS + \sigma S + \sigma_1\Sigma_1 + \sigma_2\Sigma_2 \tag{2.2}$$

and consists of potential energy due to the gravitational field of intensity g and the free energies of the interfaces Σ_1, Σ_2 and S.

N o t e . It should be pointed out that the expressions taken for the energy of the interfaces between two media are correct only if the media are quiescent. Generally speaking it does not follow that the expressions will remain the same for the case of moving media. This question is not considered here.

The Lagrangian for the system is

$$L = T - \Pi \tag{2.3}$$

and the action integral after Hamilton

$$J = \int\limits_0^{t_1} L\,dt. \tag{2.4}$$

According to Hamilton's principle the action integral for motion (2.4) assumes a stationary value, i.e. its isochronic variation vanishes: $\delta J = 0$.

We shall consider the liquid contained in the volume V as a mechanical system obeying the following relations. The flow is potential, the liquid is inextensible (capable of sustaining infinite tensions) and container walls are inpermeable, i.e.

$$\mathbf{v} = \nabla\varphi, \qquad \nabla\cdot\mathbf{v} = 0, \qquad \mathbf{v}_n = 0 \tag{2.5}$$

Also, the vertical components of the velocity of liquid particles contained in the free surface coincides with the rate of vectorial displacement of the free surface itself (*)

$$d\zeta/dt = v_z \tag{2.6}$$

The volume of the liquid does not vary, i.e. $V = \text{const}$. The last equation, isoperimetry, can be excluded if instead of considering the Lagrangian L, one considers the function

$$L' = L + \rho f V \tag{2.7}$$

where the quantity f is independent of the space variables x, y and z.

As a result of Equations (2.1) to (2.4) and (2.7) the action integral can be written in the form

$$J = \int\limits_0^{t_1} \left\{\frac{1}{2}\rho \int\limits_V v^2\,dV + \rho f V - \frac{1}{2}\rho g \int\limits_{S_0} \zeta^2 dS - \sigma S - \sigma_1\Sigma_1 - \sigma_2\Sigma_2\right\} dt \tag{2.8}$$

The problem of the description of the fluid motion can, therefore, be stated as follows: from a set of functions satisfying Equations (2.5) and (2.6) to choose functions extremizing the action integral, i.e. satisfying $\delta J = 0$.

3. We shall show that the variational statement of the problem of ideal incompressible fluid motion in fixed container and subjected to the action of the forces of gravity and surface tension, is equivalent to the classical statement of the problem.

It follows from Equations (2.5) and (2.6) that the velocity potential $\varphi(x,y,z,t)$ is a harmonic function which satisfies the following conditions:

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Sigma_2, \qquad \frac{\partial\varphi}{\partial z} = \frac{\partial\zeta}{\partial t} + \nabla\zeta\cdot\nabla\varphi \quad \text{on } S \tag{3.1}$$

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*) This condition follows from continuity of the fluid.

where n designates the external normal to the container surface.

We shall now determine the isochronic variation of the action integral (2.8). The volume V is bounded by the surface $\zeta(x,y,t)$ and is, therefore, subject to variation. In general the walls of the container can be inclined. This renders the area of S_0 also subjected to variation. Moreover, owing to the second of Equations (3.1), the variations of the functions φ and ζ on the free surface are connected by a certain relationship. Let us take variations of both sides of this equation. As the independent variables x, y and z are not subject to variation we shall get

$$\frac{\partial\delta\zeta}{\partial t} + \nabla\delta\zeta\cdot\nabla\varphi + \nabla\zeta\cdot\nabla\delta\varphi = \frac{\partial\delta\varphi}{\partial z}$$

This relationship can be written in the form

$$\frac{d\delta\zeta}{dt} = \frac{1}{\cos\gamma}\frac{\partial\delta\varphi}{\partial n}, \qquad \frac{1}{\cos\gamma} = \sqrt{1+(\nabla\zeta)^2} \tag{3.2}$$

where n denotes the outer normal to the free surface. Equation (3.2) gives the relationship between the variations $\delta\varphi$ and $\delta\zeta$ on the free surface.

Moreover, from the condition $\Delta\varphi = 0$ and the first of Equations (3.1) we find that

$$\Delta\delta\varphi = 0 \quad \text{within the volume } V, \quad \frac{\partial\delta\varphi}{\partial n} = 0 \quad \text{on the surface } \Sigma_2 \tag{3.3}$$

Now we shall evaluate the isochronic variations of the action integral (2.8)

$$\delta J = \int_0^{t_1}\Big\{\rho\int_V \nabla\varphi\cdot\nabla\delta\varphi\, dV + \frac{1}{2}\rho\int_{S_0}(\nabla\varphi)^2\delta\zeta\, dS + \frac{1}{2}\rho\int_{\delta S_0}(\nabla\varphi)^2\,\delta\zeta\, dS +$$

$$+ \rho f\delta V - \rho g\int_{S_0}\zeta\delta\zeta\, dS - \rho g\int_{\delta S_0}\zeta\delta\zeta\, dS - \sigma\delta S - \sigma_1\delta\Sigma_1 - \sigma_2\delta\Sigma_2\Big\}\, dt \tag{3.4}$$

The appearance of the second and the third terms in the last equation is due to the variations of the volume V and the area S_0. We shall discuss term by term the equation for δJ.

By Green's theorem and with the help of Equations (3.3) we shall transform the first integral thus

$$\int_V \nabla\varphi\cdot\nabla\delta\varphi\, dV = -\int_V \varphi\Delta\delta\varphi\, dV + \int_S \varphi\frac{\partial\delta\varphi}{\partial n}\, dS = \int_{S_0}\frac{1}{\cos\gamma}\varphi\frac{\partial\delta\varphi}{\partial n}\, dS$$

Now using Equation (3.2) we shall obtain

$$\int_V \nabla\varphi\cdot\nabla\delta\varphi\, dV = \int_{S_0}\varphi\frac{d\delta\zeta}{dt}\, dS \tag{3.5}$$

Leaving the second integral as is, we shall discuss the third. From Fig.2 we have

$$\int_{\delta S_0}(\nabla\varphi)^2\delta\zeta\, dS = \int_{L_0}(\nabla\varphi)^2\delta\zeta\delta l\, d\lambda$$

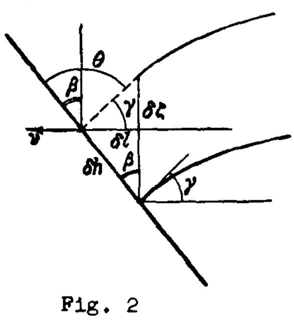

Fig. 2

where L_0 denotes the curve bounding the surface S_0. It is obvious $\delta l \sim \delta\zeta$ if the angle of the container wall inclination to the z-axis is not very small (i.e. $\tan\beta \sim 1$). The integral is therefore of order $(\delta\zeta)^2$ and can be neglected. Similarly, the sixth term can be estimated and neglected.

Let us consider the fourth term. It is easy to see that

$$\delta V = \int_{S_0} \delta\zeta\, dS + \int_{\delta S_0} \delta\zeta\, dS$$

Also, following the above argument, it can be shown that the second integral is of the order $(\delta\zeta)^2$ and can be neglected.

Hence

$$\delta V = \int_{S_0} \delta\zeta\, dS \tag{3.6}$$

Leaving the fifth term unchanged we shall examine the seventh. Clearly

$$S = \int_{S_0} \sqrt{1 + (\nabla\zeta)^2}\, dS, \qquad \delta S = \int_{S_0} \frac{\nabla\zeta\cdot\nabla\delta\zeta}{\sqrt{1+(\nabla\zeta)^2}}\, dS + \int_{\delta S_0} \frac{1}{\cos\gamma}\, dS$$

Integrating by parts it is not difficult to show that

$$\int_{S_0} \frac{\nabla\zeta\cdot\nabla\delta\zeta}{\sqrt{1+(\nabla\zeta)^2}}\, dS = \int_{L_0} \frac{\partial\zeta}{\partial\nu}\cos\gamma\delta\zeta\, d\lambda - \int_{S_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\delta\zeta\, dS$$

where ν denotes the external normal to the curve L_0, and R_1 and R_2 are the principal radii of curvature of the free surface. With this, the double mean curvature is

$$\frac{1}{R_1}+\frac{1}{R_2} = \frac{1}{[1+(\nabla\zeta)^2]^{3/2}}\left\{\Delta\zeta + \left(\frac{\partial\zeta}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\zeta}{\partial y}\frac{\partial}{\partial x}\right)^2 \zeta\right\}$$

As can be seen from Fig.2

$$\int_{\delta S_0} \frac{1}{\cos\gamma}\, dS = \int_{L_0} \frac{1}{\cos\gamma}\,\delta l\, d\lambda$$

And it is easy to show that

$$\delta l = \frac{\cos\gamma\sin\beta}{\sin\theta}\,\delta\zeta$$

Moreover $\partial\zeta/\partial\nu = -\tan\gamma$. Therefore

$$\int_{\delta S_0} \frac{1}{\cos\gamma}\, dS = \int_{L_0} \frac{\sin\beta}{\sin\theta}\,\delta\zeta\, d\lambda, \qquad \int_{L_0} \frac{\partial\zeta}{\partial\nu}\cos\gamma\delta\zeta\, d\lambda = -\int_{L_0} \sin\gamma\delta\zeta\, d\lambda$$

Finally, from Fig.2 it is clear that

$$\beta = \theta + \gamma - \tfrac{1}{2}\pi, \qquad \sin\beta = -\cos(\theta+\gamma)$$

With the help of all these formulas we find that

$$\delta S = -\int_{S_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\delta\zeta\, dS - \int_{L_0} \frac{\cos\theta\cdot\cos\gamma}{\sin\theta}\,\delta\zeta\, d\lambda \tag{3.7}$$

Let us evaluate the last two terms of Equation (3.4). Since the area of the container surface $\Sigma_1 + \Sigma_2$ remains constant, $\delta\Sigma_1 = -\delta\Sigma_2$. And, as can be seen from Fig.2

$$\delta\Sigma_2 = \int_{L_0} \delta h\, d\lambda, \qquad \delta h = \frac{\cos\gamma}{\sin\theta}\,\delta\zeta$$

Therefore

$$\sigma_1\delta\Sigma_1 + \sigma_2\delta\Sigma_2 = (-\sigma_1+\sigma_2)\int_{L_0} \frac{\cos\gamma}{\sin\theta}\,\delta\zeta\, d\lambda \tag{3.8}$$

In this way, the variation of the action integral (3.4) can be transformed with the help of Equations (3.5) through (3.8) into

$$\delta J = \int_0^{t_1} \Big\{ \rho \int_{S_0} \varphi \frac{d\delta\zeta}{dt}\, dS + \frac{1}{2}\rho \int_{S_0} (\nabla\varphi)^2\, \delta\zeta\, dS + \rho f \int_{S_0} \delta\zeta\, dS - $$

$$- \rho g \int_{S_0} \zeta\delta\zeta\, dS + \sigma \int_{S_0} \Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)\delta\zeta\, dS - \int_{L_0} (-\sigma\cos\theta - \sigma_1 + \sigma_2)\frac{\cos\gamma}{\sin\theta}\,\delta\zeta\, d\lambda \Big\}\, dt \qquad (3.9)$$

The first part of the expression can be integrated by parts with respect to time. Then assuming that $\delta\zeta = 0$ at $t = 0$ and $t = t_1$, we obtain

$$\int_0^{t_1}\int_{S_0} \varphi \frac{d\delta\zeta}{dt}\, dS\, dt = \int_{S_0} \varphi\delta\zeta \Big|_0^{t_1} dS - \int_0^{t_1}\int_S \frac{d\varphi}{dt}\,\delta\zeta\, dS\, dt = -\int_0^{t_1}\int_{S_0} \Big[\frac{\partial\varphi}{\partial t} + (\nabla\varphi)^2\Big]\delta\zeta\, dS\, dt \qquad (3.10)$$

Thus, from the condition $\delta J = 0$ and Expressions (3.9) and (3.10) it follows that

$$\int_0^{t_1} \Big\{ -\rho \int_{S_0} \Big[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\zeta - \frac{\sigma}{\rho}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) - f\Big]\, d\zeta\, dS - $$

$$- \int_{L_0} (-\sigma\cos\theta - \sigma_1 + \sigma_2)\frac{\cos\gamma}{\sin\theta}\,\delta\zeta\, d\lambda \Big\}\, dt = 0 \qquad (3.11)$$

Since the variation $\delta\zeta$ is arbitrary, it follows that with $z = \zeta(x, y, t)$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\zeta - \frac{\sigma}{\rho}\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) = f \qquad (3.12)$$

on the contour L

$$\sigma_2 = \sigma_1 + \sigma\cos\theta \qquad (3.13)$$

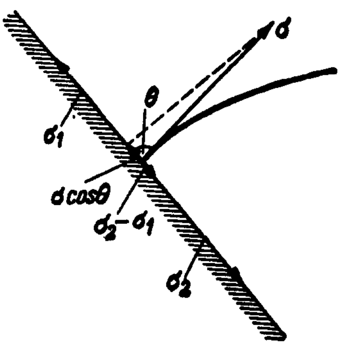

Fig. 3

Let us examine Equation (3.13). The free surface of the liquid makes an angle θ with the container wall at the point of contact. This angle is defined as a limiting angle. Equation (3.13) happens to be the condition of equilibrium at the points of contact of the three media. It can be seen from Fig.3 that, with the accepted notation, $\cos\theta > 0$, i.e. $\theta < \frac{1}{2}\pi$, in the case of convex meniscus and $\cos\theta < 0$ concave meniscus $(\theta > \frac{1}{2}\pi)$.

In this way the problem reduces to the determination of the harmonic function $\varphi(x,y,z,t)$ according to Equations (3.1), (3.12) and (3.13) for the region V.

By the same token it is shown that the variational statement of the problem of ideal, incompressible fluid motion in a container of finite dimensions is equivalent to the classical statement, by virtue of the corresponding boundary value problem.

4. Let us consider some particular cases. (a) The problem of the motion of an ideal, incompressible liquid in a container of finite dimensions is stated as follows: from the class of functions satisfying conditions (2.5) and (2.6) to find those which render $\delta J = 0$ for

$$J = \int_0^{t_1} \Big\{ \frac{1}{2}\rho \int_V (\nabla\varphi)^2\, dV - \frac{1}{2}\rho g \int_{S_0} \zeta^2\, dS + \rho f V \Big\}\, dt$$

It can be shown that these functions will satisfy

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\zeta = f \quad \text{with } z = \zeta(x, y, t)$$

This result is a direct generalization of the result obtained by Moiseev [1].

b) The search for the stationary values of the action (2.8) can be limited to a very narrow class of functions for which $\delta\zeta = 0$ on contour L. Then the variational statement of the problem of the liquid motion under the action of gravitational and surface tension forces assumes the form: from the class of functions satisfying the conditions (2.5) and (2.6) and $\delta\zeta = 0$ on the contour L, select those which satisfy condition $\delta J = 0$ for

$$J = \int_0^{t_1} \left\{ \frac{1}{2}\rho \int_V (\nabla\varphi)^2\, dV - \frac{1}{2}\rho g \int_{S_0} \zeta^2\, dS - \sigma S + \rho f V \right\} dt$$

It can be shown that these functions will satisfy

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + g\zeta - \frac{\sigma}{\rho}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = f \quad \text{with } z = \zeta(x, y, t)$$

c) The variational statement of the problem of equilibrium of a liquid enclosed in a container of finite dimensions and subjected to the action of gravitational and surface tension forces also obtains as a particular case of the discussed problem. Namely: from the class of continuous functions, select those which satisfy the condition $\delta U = 0$ for

$$U = \frac{1}{2}\rho g \int_{S_0} \zeta^2\, dS + \sigma S + \sigma_1 \Sigma_1 + \sigma_2 \Sigma_2 - \rho f V$$

It appears that these functions will satisfy

$$g\zeta - \frac{\sigma}{\rho}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = f \quad \text{with } z = \zeta(x, y)$$

$$\sigma_2 = \sigma_1 + \sigma\cos\theta \quad \text{on the contour } L \qquad (4.1)$$

If the solution of this problem is sought in a class of functions satisfying the condition $\delta\zeta = 0$ on the contour L, then

$$U = \frac{1}{2}\rho g \int_{S_0} \zeta^2\, dS + \sigma S - \rho f V$$

and the sought functions will satisfy the first of Equations (4.1).

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